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The standard complex of quantum enveloping algebras

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Abstract. We consider quantized derivation representations π of Hopf algebras $\mathcal{H}$ in some associative algebras $\mathcal{A}$. The algebra of quantized differential operators $\mathcal{A} \times_\pi \mathcal{H}$ of bialgebra representations, which are special representations by derivations, are constructed. For the quantum enveloping algebras $U_q(g)$ of Lie algebras associated with the root systems A_n, B_n, C_n and D_n we define a deformation Λ of the exterior algebra of forms, and by applying the above results it is shown that the quantized adjoint representation of $U_q(g)$ induces a bialgebra representation $\mathrm{ad}^\wedge$ of $U_q(g)$ in Λ. The resulting algebra of differential operators $\Lambda \times_{\mathrm{ad}^\wedge} U_q(g)$ is a deformation of the standard Koszul complex of Lie algebras. It admits an exterior derivative which is, in particular, a $U_q(g)$-module morphism. Hence cohomology groups of $U_q(g)$ relative to some $U_q(g)$-module $\mathcal{M}$ can be constructed.

1. Introduction

For several kinds of quantum groups bicovariant differential calculi were developed [CSWW, Jur, Wor]. They provide higher-order differential calculi $\Gamma^\wedge$ with an exterior derivative $\mathcal{D}$ so that $(\Gamma^\wedge, \mathcal{D})$ is a complex. Especially for the quantum groups of Lie algebras g of type A_n, B_n, C_n and D_n the differential calculi, where the algebra generated by the (left-invariant) vector fields is equivalent to the quantum enveloping algebra $U_q(g)$, are of particular interest [CSWW, Jur]—the complex $(\Gamma^\wedge, \mathcal{D})$ can then be considered as a deformation of the de Rham complex of the corresponding Lie group. The resulting cohomology groups are in general different from the classical ones [Gri].

In the classical theory besides the de Rham cohomology of Lie groups a cohomology structure for Lie algebras exists which is initiated by the so-called standard or Koszul complex of Lie algebras [CE, HS, Jac]. Key ingredients for the explicit construction are the exterior algebra over the Lie algebra, the adjoint representation of the Lie algebra and the induced algebra of differential operators [Jac].

The aim of this paper is the construction of a complex over the quantum enveloping algebra $U_q(g)$ for Lie algebras g of type A_n, B_n, C_n and D_n which yields a homological structure and admits a (dual, $U_q(g)$-modular) cohomology. This complex turns out to be a

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deformation of the classical Koszul complex for Lie algebras. We show that all essential results of the classical theory can be generalized to the quantum case. Especially the nilpotent derivative d has analogous properties. Though there are differences between the classical and the standard complex. Namely the quantum Koszul complex is no more exact and it seems therefore that no categorial definition is possible for the cohomologies of $U_q(g)$, a fact which already appeared in the construction of differential calculi of quantum groups. Nevertheless, there is some uniqueness for the homology and cohomology structures of $U_q(g)$ since vector fields are used which generate $U_q(g)$—we conjecture the independence of the cohomologies of the different $U_q(g)$-generating sets of vector fields.

For a better understanding of the paper we presuppose knowledge of Hopf algebras and quantum groups [Abe, Dri, FRT]. Section 2 begins with an introduction to basic concepts needed in the course of our work like, for instance, the representation of Hopf algebras by quantized derivations and the algebra of differential operators. At the end of section 2 the quantized (left-)adjoint representation is presented. The third section is almost entirely dedicated to the construction and investigation of the q-exterior algebra of forms which is, besides the quantum enveloping algebra, one of the components to build up the quantum standard complex. This is defined and studied in section 4. The main result of section 4 is theorem 4.2 which shows the analogies of the quantum and the classical standard complex for Lie algebras. The cohomology structure induced by the quantum Koszul complex is exhibited and the non-exactness of this complex is explicitly exposed. Section 4 ends with a brief discussion of modifications of the quantum Koszul complex where the sources of non-exactness are removed and which therefore are shown to be exact. Finally, we refer the reader to the appendix where we recall essential definitions and results [CDSWZ, CSWW, DJSWZ, DSWZ, FRT, Res, Wor] needed especially for the studies in sections 3 and 4 and partially for the investigations in section 2.

2. The algebra of quantized differential operators

We define representations of a Hopf algebra $\mathcal{H}$ by quantized derivations in an algebra† $\mathcal{A}$. We denote by Δ, ϵ and S the comultiplication, the counit and the antipode of $\mathcal{H}$, respectively.

Definition 2.1. A representation π of a Hopf algebra $\mathcal{H}$ by quantized derivations in an algebra $\mathcal{A}$ is an algebra homomorphism

$$\pi : U_q(g) \to \mathrm{End}(\mathcal{A})$$

with the additional property

$$\pi(v)(a\,b) = \pi(v_{(1)})(a)\pi(v_{(2)})(b) \qquad \forall v \in \mathcal{H} \quad a, b \in \mathcal{A} \tag{2.1}$$

where $\Delta(v) = v_{(1)} \otimes v_{(2)}$ is the notation of the coproduct of v [Abe].

Especially for the quantum enveloping algebras $U_q(g)$ of Lie algebras g of type A_n, B_n, C_n and D_n we can now formulate

† When we speak of (vector) spaces and algebras we always mean $\mathbb{C}$-vector spaces and associative, unital $\mathbb{C}$-algebras, respectively.

Proposition 2.1. Let π be a representation of $U_q(g)$ by derivations in an algebra $\mathcal{A}$ and let $\pi(\mathbf{1}_{U_q(g)}) = \mathrm{id}_{\mathcal{A}}$. Then for all vector fields $X \in T \subset U_q(g)$ the derivation vanishes at the unity of $\mathcal{A}$, i.e.

$$\pi(X)(\mathbf{1}_{\mathcal{A}}) = 0 \qquad \forall X \in T \tag{2.2}$$

or equivalently

$$\pi(v)(\mathbf{1}_{\mathcal{A}}) = \epsilon(v)\,\mathbf{1}_{\mathcal{A}} \qquad \forall v \in U_q(g)\,. \tag{2.3}$$

Proof. We label the vector fields (X^i_j) by capital indices I, i.e. (X_I). Equation (A.5) can then be written as

$$\Delta(X_I) = X_I \otimes \mathbf{1}_{U_q(g)} + \Theta_{IJ} \otimes X_J \tag{2.4}$$

where $\Theta_{IJ} \in U_q(g)$ obeys the relations

$$\Delta(\Theta_{IJ}) = \Theta_{IK} \otimes \Theta_{KJ} \qquad \epsilon(\Theta_{IJ}) = \delta_{IJ}\,. \tag{2.5}$$

Since π is representation by derivations we have

$$\begin{aligned} &\pi(X_I)(b) = \pi(X_I)(b) + \pi(\Theta_{IJ})(b)\,\pi(X_J)(\mathbf{1}_{\mathcal{A}}) \qquad \forall b \in \mathcal{A} \\ \Longleftrightarrow\; &\pi(\Theta_{IJ})(\cdot)\,\pi(X_J)(\mathbf{1}_{\mathcal{A}}) = 0 \end{aligned}$$

and this yields

$$\begin{aligned} \pi(X_I)(\mathbf{1}_{\mathcal{A}}) &= \pi(\Theta_{KJ}\,S^{-1}(\Theta_{IK}))(\mathbf{1}_{\mathcal{A}})\,\pi(X_J)(\mathbf{1}_{\mathcal{A}}) \\ &= \pi(\Theta_{KJ})(\pi(S^{-1}(\Theta_{IK}))(\mathbf{1}_{\mathcal{A}}))\,\pi(X_J)(\mathbf{1}_{\mathcal{A}}) \\ &= 0\,. \end{aligned}$$

Equation (2.3) follows since the vector fields generate $U_q(g)$ and since $\pi(\mathbf{1}_{U_q(g)}) = \mathrm{id}_{\mathcal{A}}$. □

Remark 1. For the construction in section 3 one has to use the vector fields as a generating system to assure the closure of the one-forms under the adjoint action (see proposition 3.1).

Remark 2. Every representation of $U_q(g)$ by derivations in an algebra $\mathcal{A}$ is thus a bialgebra representation of $U_q(g)$ in $\mathcal{A}$, i.e. $\pi(v)(\mathbf{1}_{\mathcal{A}}) = \epsilon(v)\,\mathbf{1}_{\mathcal{A}}\ \forall v \in U_q(g)$ [Abe].

Now let L be the representation of a Hopf algebra $\mathcal{H}$ by left multiplication

$$L(v)(w) = v\,w \qquad \forall v, w \in \mathcal{H} \tag{2.6}$$

and $L_{\mathcal{A}}$ be the representation of $\mathcal{A}$ in $\mathcal{A} \otimes \mathcal{H}$ by left multiplication,

$$L_{\mathcal{A}}(a)(b_i \otimes v_i) := (a\,b_i) \otimes v_i \qquad \forall a \in \mathcal{A} \quad b_i \otimes v_i \in \mathcal{A} \otimes \mathcal{H}\,. \tag{2.7}$$

Given a representation π of $\mathcal{H}$ by derivations in $\mathcal{A}$ we can combine it with the representation L to a representation of $\mathcal{H}$ in the tensor space $\mathcal{A} \otimes \mathcal{H}$. We define

$$L_{\mathcal{H}} := (\pi \otimes L)\,\Delta : \mathcal{H} \to \mathrm{End}(\mathcal{A} \otimes \mathcal{H})\,. \tag{2.8}$$

Both $L_{\mathcal{A}}(a)$ and $L_{\mathcal{H}}(v)$ for fixed $a \in \mathcal{A}$ and $v \in \mathcal{H}$ are elements of $\mathrm{End}(\mathcal{A} \otimes \mathcal{H})$. In the following we investigate the subalgebra $\mathcal{S}_\pi$ of $\mathrm{End}(\mathcal{A} \otimes \mathcal{H})$ generated by the $L_{\mathcal{A}}(a)$ and $L_{\mathcal{H}}(v)$ ($a \in \mathcal{A}$, $v \in \mathcal{H}$) and use the notation $l(a) := L_{\mathcal{A}}(a)$, $l(v) := L_{\mathcal{H}}(v)$ $\forall a \in \mathcal{A} v \in \mathcal{H}$.

$$\mathcal{S}_\pi := \langle (l(a) l(v)) \,|\, a \in \mathcal{A} v \in \mathcal{H} \rangle \,. \tag{2.9}$$

Proposition 2.2. Let π be a representation of the Hopf algebra $\mathcal{H}$ by derivations in an algebra $\mathcal{A}$ and let $\mathcal{S}_\pi$ be the algebra according to (2.9). Then $\mathcal{S}_\pi$ has the following properties.

(i) The algebra $\mathcal{S}_\pi$ is unital.

(ii)

$$l(v)\, l(a) = l(\pi(v_{(1)})(a))\, l(v_{(2)}) \qquad \forall a \in \mathcal{A} \quad v \in \mathcal{H} \tag{2.10}$$

and every element $s \in \mathcal{S}_\pi$ can be written in the form

$$s = \sum_i l(a_i)\, l(v_i) \tag{2.11}$$

where $a_i \in \mathcal{A}$ and $v_i \in \mathcal{H}$.

(iii) If π is a bialgebra representation of $\mathcal{H}$ in $\mathcal{A}$ then

$$l(a)\, l(S(v)) = l(S(v_{(1)}))\, l(\pi(v_{(2)})(a)) \qquad \forall a \in \mathcal{A} \quad v \in \mathcal{H}\,. \tag{2.12}$$

Proof. Since $\mathcal{A}$ and $\mathcal{H}$ are unital we have $l(\mathbf{1}_{\mathcal{A}}) = l(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{S}_\pi}$. We apply $l(v)\, l(a)$ (where $v \in \mathcal{H}$, $a \in \mathcal{A}$) to an element $b_i \otimes u_i \in \mathcal{A} \otimes \mathcal{H}$ and exploit the derivation properties of the representation π

$$\begin{aligned} l(v)\, l(a)(b_i \otimes u_i) &= \pi(v_{(1)})(a)\, \pi(v_{(2)})(b_i) \otimes v_{(3)} u_i \\ &= l(\pi(v_{(1)})(a))\, l(v_{(2)})(b_i \otimes u_i)\,. \end{aligned}$$

Now the relation in (2.12) is an immediate consequence if we use the Hopf identities $m\,(S \otimes \mathrm{id})\,\Delta = m\,(\mathrm{id} \otimes S)\,\Delta = \eta \circ \epsilon$. The structure of the commutation relations in (2.10) admits an ordering like in (2.11) for every $s \in \mathcal{S}_\pi$. □

Proposition 2.2 points to a close relation between the vector spaces $\mathcal{A} \otimes \mathcal{H}$ and $\mathcal{S}_\pi$. This is described in the following proposition.

Proposition 2.3. Let $\mathcal{H}$, $\mathcal{A}$, π and S_π be like in proposition 2.2. Then there exists a vector space epimorphism ψ_π such that

$$\psi_\pi : \begin{cases} \mathcal{A} \otimes \mathcal{H} \to \mathcal{S}_\pi \\ \sum_i a_i \otimes v_i \mapsto \sum_i l(a_i)\, l(v_i)\,. \end{cases} \tag{2.13}$$

If π is a bialgebra representation of $\mathcal{H}$ in $\mathcal{A}$ then ψ_π is an isomorphism.

Proof. Obviously the mapping ψ_π is a bilinear vector space homomorphism. It is surjective because of (2.11). If π is a bialgebra representation then

$$l(u)(\mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{A}} \otimes u \qquad \forall u \in \mathcal{H}$$

and thus

$$l(a)\, l(u)(\mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{\mathcal{H}}) = a \otimes u \qquad \forall a \in \mathcal{A} \quad u \in \mathcal{H}\,.$$

Hence ψ_π is injective. □

Since $\mathcal{S}_\pi$ is a unital algebra and $\mathcal{S}_\pi \overset{\text{vec}}{\cong} \mathcal{A} \otimes \mathcal{H}$ if π is a bialgebra representation we can induce an algebra structure on $\mathcal{A} \otimes \mathcal{H}$ with the help of the mapping ψ_π according to

$$x * y := \psi_\pi^{-1}(\psi_\pi(x)\, \psi_\pi(y)) \tag{2.14}$$

which makes $(\mathcal{A} \otimes \mathcal{H} *)$ a unital algebra [Swe].

Remark 1. The product '$*$' in $(\mathcal{A} \otimes \mathcal{H}, *)$ is sometimes called a smash (or semi-direct) product [Swe]. It does not coincide with the usual tensor multiplication m for the tensor product of the two algebras $\mathcal{A}$ and $\mathcal{H}$ given by $m = (m_\mathcal{A} \otimes m_\mathcal{H})(\mathrm{id}_\mathcal{A} \otimes \tau \otimes \mathrm{id}_\mathcal{H})$.

Remark 2. As in sections 3 and 4 we work exclusively with the quantum enveloping algebra $U_q(g)$ and since for $U_q(g)$ every representation by derivations is a bialgebra representation (see proposition 2.1) we restrict our considerations to bialgebra representations henceforth.

Definition 2.2. The algebra $(\mathcal{A} \otimes \mathcal{H}, *)$ is called the algebra of quantized differential operators of the bialgebra representation π of $\mathcal{H}$ and is denoted by $\mathcal{A} \times_\pi \mathcal{H}$.

The following lemma exhibits useful properties of algebras of differential operators.

Lemma 2.4. Let π be a bialgebra representation of the Hopf algebra $\mathcal{H}$ in the algebra $\mathcal{A}$ and let $\mathcal{A} \times_\pi \mathcal{H}$ be the corresponding algebra of differential operators. Then

(i)

$$\mathcal{A} \overset{\text{alg}}{\cong} \{a \otimes \mathbf{1}_\mathcal{H} | a \in \mathcal{A}\} \overset{\text{alg}}{\leqslant} \mathcal{A} \times_\pi \mathcal{H}$$

$$\mathcal{H} \overset{\text{alg}}{\cong} \{\mathbf{1}_\mathcal{A} \otimes v | v \in \mathcal{H}\} \overset{\text{alg}}{\leqslant} \mathcal{A} \times_\pi \mathcal{H}$$

and we can identify $a = a \otimes \mathbf{1}_\mathcal{H}$ and $v = \mathbf{1}_\mathcal{A} \otimes v$ $\forall a \in \mathcal{A},\ v \in \mathcal{H}$.

(ii) This identification yields

$$a * v = a \otimes v \ \forall a \in \mathcal{A} \quad v \in \mathcal{H}$$

and every $x \in \mathcal{A} \times_\pi \mathcal{H}$ can be written in the form

$$x = \sum_i a_i * v_i \tag{2.15}$$

where $a_i \in \mathcal{A},\ v_i \in \mathcal{H}$. Thus

$$x = \sum_i a_i * v_i = 0 \Longleftrightarrow \sum_i a_i \otimes v_i = 0\,.$$

(iii) For the 'mixed' commutation relations one finds $\forall v \in \mathcal{H},\ a \in \mathcal{A}$

$$\begin{aligned} &v * a = \pi(v_{(1)})(a) * v_{(2)} \\ &\text{or} \quad a * S(v) = S(v_{(1)}) * \pi(v_{(2)})(a)\,. \end{aligned} \tag{2.16}$$

Proof. The injections $i_\mathcal{A} : \mathcal{A} \ni a \mapsto a \otimes \mathbf{1}_\mathcal{H} \in \mathcal{A} \times_\pi \mathcal{H}$ and $i_\mathcal{H} : \mathcal{H} \ni v \mapsto \mathbf{1}_\mathcal{A} \otimes v \in \mathcal{A} \times_\pi \mathcal{H}$ are algebra homomorphisms. This can be seen if we use the multiplication rule (2.14) for $\mathcal{A} \times_\pi \mathcal{H}$. With the help of the same rule we obtain $\forall a \in \mathcal{A},\ v \in \mathcal{H}$

$$\begin{aligned} a * v &= (a \otimes \mathbf{1}_\mathcal{H})\,(\mathbf{1}_\mathcal{A} \otimes v) \\ &= \psi_\pi^{-1}(l(a)\, l(v)) \\ &= a \otimes v\,. \end{aligned}$$

Since the mapping ψ_π is an algebra isomorphism the remaining part of lemmas 2.4.2 and 2.4.3 will follow immediately with propositions 2.2.2 and 2.2.3. $\square$

Finally, we consider the quantized (left-)adjoint representation of a Hopf algebra $\mathcal{H}$ in a $\mathcal{H}$-bimodule $\mathcal{B}$. It is defined through

$$\mathrm{ad}: \begin{cases} \mathcal{H} \to \mathrm{End}(\mathcal{B}) \\ v \mapsto \mathrm{ad}(v) := v_{(1)}\,(\cdot)\,S(v_{(2)})\,. \end{cases} \tag{2.17}$$

Lemma 2.5. If $\mathcal{B}$ is an algebra then for the Hopf algebra $\mathcal{H}$ the quantized adjoint representation ad is a bialgebra representation of $\mathcal{H}$ in $\mathcal{B}$.

Proof. Using the Hopf properties $m(S \otimes \mathrm{id})\Delta = \eta \circ \epsilon$ and $(\mathrm{id} \otimes \epsilon)\Delta = \mathrm{id}$ we find immediately $\forall u \in \mathcal{H}, v, w \in \mathcal{B}$

$$\begin{aligned} \mathrm{ad}(u)(v\,w) &= u_{(1)}\,v\,S(u_{(2)})\,u_{(3)}\,w\,S(u_{(4)}) \\ &= \mathrm{ad}(u_{(1)})(v)\,\mathrm{ad}(u_{(2)})(w) \end{aligned}$$

and $\mathrm{ad}(v)(\mathbf{1}_{\mathcal{B}}) = \epsilon(v)\,\mathbf{1}_{\mathcal{B}}$. Hence ad is a representation by derivations. □

3. The q-exterior algebra of forms

For the quantum enveloping algebras $U_q(g)$ of Lie algebras g of type A_n, B_n, C_n and D_n the quantized adjoint representation is the q-analogue of the adjoint representation of the Lie algebra g or of the corresponding universal enveloping algebra $U(g)$ [Jur, Wor]. The quantized adjoint representation will play a fundamental role in the construction of a special algebra of differential operators which turns out to be a deformation of the classical standard complex (or Koszul complex) of Lie algebras. The deformed complex is built of the q-exterior algebra of forms and the quantized enveloping algebra. Before we define the q-exterior algebra of forms we exhibit some properties of the adjoint representation of $U_q(g)$.

Proposition 3.1. The vector space of vector fields T of the quantum enveloping algebra $U_q(g)$ is closed under the adjoint representation, i.e.

$$\mathrm{ad}(u)(X) \in T \qquad \forall u \in U_q(g),\ X \in T\,. \tag{3.1}$$

Proof. With the help of (A.1) one finds

$$\mathrm{ad}(L^{\pm i}{}_j)(X^k{}_l) = L^{\pm i}{}_r X^k{}_l S(L^{\pm r}{}_l)\,.$$

The application of some $(\varepsilon, \hat{R}_q)$- or $(C, \hat{R}_q)$-identities [CSWW, FRT, Wor] and of (A.2) then leads to

$$\mathrm{ad}(L^{\pm i}{}_j)(X^k{}_l) = (\hat{R}_q^{\mp 1ki}{}_{cb}\,\hat{R}_q^{\pm 1ca}{}_{lj})\,X^b{}_a \tag{3.2}$$

and the statement of the proof follows since ad is representation and $U_q(g)$ is generated by the $L^{\pm i}{}_j$. □

Remark 1. Similar to [CSWW, Jur] it can be shown that $\hat{T} := S^{-1}(T)$ is the corresponding vector space dual to the space of left invariant elements of a bicovariant first-order differential calculus [Wor]. Exploiting the results of [Wor] the statement of proposition 3.1 follows as an immediate consequence.

Remark 2. The quantized Lie bracket

$$[X, Y] := \mathrm{ad}(X)(Y) \qquad \forall X,\ Y \in T \tag{3.3}$$

is a bilinear product in T [Wor].

Now we perform a basis transformation in T according to (A.7) and (A.8).

$$X^i{}_j \mapsto \begin{cases} X^{i[j]} & \text{for} \quad A_n \\ X^{ij} & \text{for} \quad B_n,\ C_n,\ D_n\,. \end{cases} \tag{3.4}$$

For A_n a direct calculation yields on the one hand

$$\mathrm{ad}(X^{k[l]})(X^{i[j]}) = X^{k[l]}\, X^{i[j]} - \mathrm{ad}(\Theta^k{}_{[b]}{}^{[l]}{}_a)(X^{i[j]})\, X^{a[b]} \tag{3.5}$$

and on the other hand

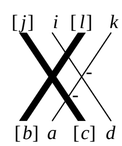
$$\mathrm{ad}(X^{k[l]})(X^{i[j]}) = \frac{1}{\lambda}\,(C^{[l]k}\, X^{i[j]} - C^{[b]a}\,\mathrm{ad}(\Theta^k{}_{[b]}{}^{[l]}{}_a)(X^{i[j]}))\,. \tag{3.6}$$

We used equations (A.4) and (A.5) for the vector fields. To evaluate $\mathrm{ad}(\Theta^k{}_{[b]}{}^{[l]}{}_a)(X^{i[j]})$ we need the explicit expression of the Θ's in terms of the $L^{\pm i}{}_j$ (see equation (A.5) below) and the identity (3.2). Making use of several $(\varepsilon, \hat{R}_q)$-relations we arrive, after some computation, at

$$\mathrm{ad}(\Theta^k{}_{[b]}{}^{[l]}{}_a)(X^{i[j]}) = \hat{\sigma}^{[j]i[l]k}{}_{[b]a[c]d}\, X^{d[c]} \tag{3.7}$$

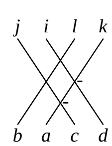
where

$$\hat{\sigma}^{[j]i[l]k}{}_{[b]a[c]d} = q^{-(n-1)} \tag{3.8}$$



(graphical presentation of the matrix of the adjoint representation of Θ acting on the vector fields $X^i{}_j$ in the case A_n) according to the graphical representation (A.11) and (A.12). Similar results are obtained for B_n, C_n and D_n (graphical presentation of the matrix of the adjoint representation of Θ acting on the vector fields$X^i{}_j$ in the case B_n, C_n and D_n) where

$$\hat{\sigma}^{jilk}{}_{bacd} = \quad . \tag{3.9}$$



Henceforth we denote by $\sigma : T \otimes T \to T \otimes T$ the linear mapping corresponding to its matrix representation (3.8) for A_n or (3.9) for B_n, C_n, D_n and use the notation $\sigma\,(X \otimes Y) := Y_{(s)} \otimes X_{(s)}\ \forall X,\ Y \in T$.

Corollary 3.2. Let g be a Lie algebra of type A_n. Then the quantized Lie bracket

$$[X, Y] = X\,Y - Y_{(s)}\,X_{(s)} \qquad \forall X,\ Y \in T$$

takes the form

$$\begin{aligned}[][X^{i[j]}, X^{k[l]}] &= X^{i[j]}\,X^{k[l]} - \hat{\sigma}^{[l]k[j]i}{}_{[b]a[c]d}\,X^{d[c]}\,X^{a[b]} \\ &= \frac{1}{\lambda}\left(C^{[j]i}\,X^{k[l]} - \hat{\sigma}^{[l]k[j]i}{}_{[b]a[c]d}\,C^{[b]a}\,X^{d[c]}\right) \end{aligned} \tag{3.10}$$

for the basis $\{X^{i[j]}\} \subset T \subset U_q(g)$. In the cases of B_n, C_n and D_n analogous results hold. □

This suggests how to define a q-deformation of the exterior algebra over a Lie algebra.

Definition 3.1. The algebra

$$\Lambda(T) := \bigoplus_{m=0}^{\infty} T^{\otimes m} \Big/ (\mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma)) \tag{3.11}$$

will be called a quantized or q-exterior algebra of the quantum enveloping algebra $U_q(g)$.

Since $\Lambda(T)$ is obtained by factorizing out a homogenous ideal, i.e.

$$I = (\mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma)) = \bigoplus_{m=0}^{\infty} (I \cap T^{\otimes m})$$

we find at once

$$\Lambda(T) = \bigoplus_{m=0}^{\infty} \Lambda_m(T) \tag{3.12}$$

where $\Lambda_m(T)$ is the space of monomials in T of degree m and $\Lambda_0(T) = \mathbb{C}$. In section 4 we will see that after introducing a derivative d in the algebra of differential operators $\Lambda(T) \times_{\mathrm{ad}^\wedge} U_q(g)$, where $\mathrm{ad}^\wedge$ is defined in proposition 3.4, the definition for the q-exterior product follows immediately. This can be seen explicitly in the proof of theorem 4.2.

Before investigating extensions of the adjoint representation of $U_q(g)$ to the algebra $\Lambda(T)$ we analyse the mapping σ. Similar as in [CSWW] one realizes

Proposition 3.3. The linear mapping $\sigma : T \otimes T \to T \otimes T$ is bijective and obeys the braid group equation

$$(\sigma \otimes \mathrm{id}_T)(\mathrm{id}_T \otimes \sigma)(\sigma \otimes \mathrm{id}_T) = (\mathrm{id}_T \otimes \sigma)(\sigma \otimes \mathrm{id}_T)(\mathrm{id}_T \otimes \sigma)\,. \tag{3.13}$$

It admits a complete projector decomposition according to its minimal polynomial

$$(\sigma - \mathrm{id}_{T^{\otimes 2}})(\sigma + q^2\,\mathrm{id}_{T^{\otimes 2}})(\sigma + q^{-2}\,\mathrm{id}_{T^{\otimes 2}}) = 0 \tag{3.14}$$

for A_n and

$$\begin{aligned}&(\sigma - \mathrm{id}_{T^{\otimes 2}})(\sigma + q^2\,\mathrm{id}_{T^{\otimes 2}})(\sigma + q^{-2}\,\mathrm{id}_{T^{\otimes 2}})(\sigma + \varepsilon\,q^{\varepsilon-N+1}\,\mathrm{id}_{T^{\otimes 2}}) \\ &\quad \times (\sigma + \varepsilon\,q^{N-\varepsilon-1}\,\mathrm{id}_{T^{\otimes 2}})(\sigma - \varepsilon\,q^{\varepsilon-N-1}\,\mathrm{id}_{T^{\otimes 2}})(\sigma - \varepsilon\,q^{N-\varepsilon+1}\,\mathrm{id}_{T^{\otimes 2}}) = 0\end{aligned} \tag{3.15}$$

for B_n, C_n and D_n.

Proof. The matrices $\hat{\sigma}$ in (3.8) and (3.9) are invertible. Since the matrices $\hat{R}_q$ obey the braid group equation we can conclude from the structure of $\hat{\sigma}$ that it also satisfies the braid group equation. Using the projector decomposition of the matrices $\hat{R}_q$ and $\hat{\mathcal{R}}_q$ given by (A.13) for A_n or the decomposition of $\hat{R}_q$ given by (A.14) for B_n, C_n and D_n one easily proves equations (3.14) and (3.15). □

Proposition 3.4. For a Lie algebra g of type A_n, B_n, C_n or D_n let $\Lambda(T)$ be the q-exterior algebra of $U_q(g)$ according to definition 3.1. Then there exists a unique mapping

$$\mathrm{ad}^{\wedge} : U_q(g) \to \mathrm{End}(\Lambda(T))$$

such that

(i) $\mathrm{ad}^{\wedge}(u)_{|T} = \mathrm{ad}(u)_{|T}\ \forall u \in U_q(g)$,

(ii) $\mathrm{ad}^{\wedge}$ is a representation of $U_q(g)$ by derivations.

Proof. In a frist step we define the mapping

$$\begin{aligned} &\mathrm{ad}^{\otimes} : U_q(g) \to \mathrm{End}(\mathcal{T}) \\ &\mathcal{T} := \bigoplus_{m=0}^{\infty} T^{\otimes m} \end{aligned} \tag{3.16}$$

through

$$\mathrm{ad}^{\otimes} = \bigoplus_{m=0}^{\infty} \mathrm{ad}_m^{\otimes} := \epsilon \oplus \bigoplus_{m=1}^{\infty} (\underbrace{\mathrm{ad} \otimes \cdots \otimes \mathrm{ad}}_{m\ \text{times}})\, \Delta^{(m-1)}$$

where

$$\Delta^{(m)} = \begin{cases} (\Delta \otimes \underbrace{\mathrm{id}_{U_q(g)} \otimes \cdots \otimes \mathrm{id}_{U_q(g)}}_{(m-1)\ \text{times}}) \cdot \cdots \cdot \Delta & \text{for} \quad m > 0 \\ \mathrm{id}_{U_q(g)} & \text{for} \quad m = 0\,. \end{cases}$$

One immediately verifies that $\mathrm{ad}^{\otimes}$ is a representation of $U_q(g)$ by derivations in $\mathcal{T}$. In the next step we shall make use of

Lemma 3.5. For any ideal $I \subset \mathcal{T}$ such that

$$\mathrm{ad}^{\otimes}(u)(I) \subset I \qquad \forall u \in U_q(g)$$

the canonically induced mapping

$$\overline{\mathrm{ad}^{\otimes}} : U_q(g) \to \mathrm{End}(\mathcal{T}/I)$$

is a representation of $U_q(g)$ by derivations in $\mathcal{T}/I$.

Proof. For all $u \in U_q(g)$, $\bar{t} \in \mathcal{T}/I$ we have $\overline{\mathrm{ad}^{\otimes}}(u)(\bar{t}) = \overline{\mathrm{ad}^{\otimes}(u)(t)}$ for any $t \in \bar{t} \subset \mathcal{T}$. Hence

$$\overline{\mathrm{ad}^{\otimes}}(u)(\bar{t}\,\bar{t'}) = \overline{\mathrm{ad}^{\otimes}}(u_{(1)})(\bar{t})\,\overline{\mathrm{ad}^{\otimes}}(u_{(2)})(\bar{t'})\,.$$

The representation properties are evident. □

To prove the statement of the proposition it is sufficient to show that

$$\mathrm{ad}^{\otimes}(X)(\mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma)) \subset \mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma) \qquad \forall X \in T$$

since $\mathrm{ad}^{\otimes}$ is representation and T generates $U_q(g)$. From proposition 3.3 we conclude

$$\mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma) = \mathrm{Im}(P_1(\sigma))$$

where the projector $P_1(\sigma)$ is given by

$$P_1(\sigma) = \mathcal{N}\,(\sigma + q^2\,\mathrm{id}_{T^{\otimes 2}})(\sigma + q^{-2}\,\mathrm{id}_{T^{\otimes 2}}) \tag{3.17}$$

for A_n and

$$\begin{aligned} P_1(\sigma) = \mathcal{N}'(\sigma + q^2\,\mathrm{id}_{T^{\otimes 2}})&(\sigma + q^{-2}\,\mathrm{id}_{T^{\otimes 2}})(\sigma + \varepsilon\, q^{\varepsilon-N+1}\,\mathrm{id}_{T^{\otimes 2}}) \\ &\times (\sigma + \varepsilon\, q^{N-\varepsilon-1}\,\mathrm{id}_{T^{\otimes 2}})(\sigma - \varepsilon\, q^{\varepsilon-N-1}\,\mathrm{id}_{T^{\otimes 2}})(\sigma - \varepsilon\, q^{N-\varepsilon+1}\,\mathrm{id}_{T^{\otimes 2}}) \end{aligned} \tag{3.18}$$

for B_n, C_n and D_n. $\mathcal{N}$ and $\mathcal{N}'$ are normalization constants.

In the case of A_n let

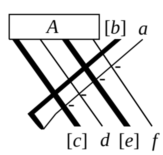
$$\begin{aligned} & X = A_{[i]j[k]l}\, X^{l[k]}\, X^{j[i]} \in \mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma) \\ \Longleftrightarrow\; & X \in \mathrm{Im}(P_1(\sigma)) \\ \Longleftrightarrow\; & A_{[i]j[k]l} = (B \cdot P_1(\hat{\sigma}))_{[i]j[k]l} \end{aligned}$$

with some row vector B. Now apply $(\mathrm{id}_{T^{\otimes 2}} - \sigma)$ to $\mathrm{ad}^{\otimes}(X^{a[b]})(X)$:

$$\begin{aligned} (\mathrm{id}_{T^{\otimes 2}} - \sigma)\,&\mathrm{ad}^{\otimes}(X^{i[j]})(X) \\ &= \frac{1}{\lambda}\, A_{[c]d[e]f}\big(C^{[b]a}\,\delta^{[c]d[e]f}{}_{[m]l[u]v} - \hat{\sigma}^{[e]f[b]a}{}_{[g]h[u]v}\,\hat{\sigma}^{[c]d[g]h}{}_{[r]s[m]l}\,C^{[r]s}\big) \\ &\quad \times (\mathrm{id}_{T^{\otimes 2}} - \sigma)(X^{v[u]} \otimes X^{l[m]}) \end{aligned}$$

where we used the relation (3.10). We evaluate the action of the linear map $(\mathrm{id}_{T^{\otimes 2}} - \sigma)$ on $(X^{v[u]} \otimes X^{l[m]})$ and use the fact that $A \cdot (\mathbf{1} - \hat{\sigma}) = 0$. Then we obtain:

$$\begin{aligned} (\mathrm{id}_{T^{\otimes 2}} - \sigma)\,\mathrm{ad}^{\otimes}(X^{i[j]})(X) &= -\frac{1}{\lambda}\,(A_{[g]h[i]j}\,\hat{\sigma}^{[i]j[b]a}{}_{[k]l[e]f}\,\hat{\sigma}^{[g]h[k]l}{}_{[m]n[c]d}\,C^{[m]n}) \\ &\quad \times (\mathbf{1} - \hat{\sigma})^{[c]d[e]f}{}_{[r]s[t]u}\,X^{u[t]} \otimes X^{s[r]} \end{aligned}$$

$\overset{\text{graphics}}{=}$ 

$(\mathbf{1} - \hat{\sigma})^{[c]d[e]f}{}_{[r]s[t]u}\,X^{u[t]} \otimes X^{s[r]}$

(graphical presentation of the matrix of the adjoint representation of the vector fields $X^i{}_j$ acting on an element X of the kernel of $(\mathrm{id} - \sigma)$ in the case A_n). Since $\hat{\sigma}$ obeys the braid group relation (3.13), $P_1(\hat{\sigma})$ is a polynomial in $\hat{\sigma}$ and $A = (B \cdot P_1(\hat{\sigma}))$ for some row vector B we obtain

$$\mathrm{ad}^{\otimes}(X^{i[j]})(X) \in \mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma)\,.$$

Analogous results hold for B_n, C_n and D_n. Together with lemma 3.5 this concludes the proof. $\square$

4. Cohomology of quantum enveloping algebras

We are now in a position to build a complex which yields the deformed homology and cohomology structures on $U_q(g)$. Let g be a Lie algebra of type A_n, B_n, C_n or D_n. Then we can construct the algebra of differential operators of the adjoint representation $\mathrm{ad}^\wedge$ of $U_q(g)$ in $\Lambda(T)$:

$$K(q, g) := \Lambda(T) \times_{\mathrm{ad}^\wedge} U_q(g) \,. \tag{4.1}$$

There arise slight complications in the notation since $K(q, g)$ contains two different copies of T, one in $U_q(g)$ and the other in $\Lambda(T)$. Therefore we denote the vector space of vector fields contained in $U_q(g)$ by $\tilde{T}$ and the elements of $\tilde{T}$ by $\tilde{X}$, $\tilde{Y}, \ldots$.

Proposition 4.1. (i) The algebra $K(q, g)$ is a graded algebra.

$$K(q, g) = \bigoplus_{m=0}^{\infty} K_m(q, g) \tag{4.2}$$

where $K_m(q, g) = \Lambda_m(T) * U_q(g)$.

(ii) In $K(q, g)$ there exists a unique grade indicating algebra isomorphism γ such that $\gamma\,(k_m) = (-1)^m k_m \forall k_m \in K_m(q, g)$.

Proof. *Ad 1*. Let $X \in T \subset \Lambda(T)$, $v \in U_q(g)$. Then lemma 2.4.3 and proposition 3.1 yield

$$v * X = \mathrm{ad}^\wedge(v_{(1)})(X) * v_{(2)} = X'_{(1)} * v_{(2)}$$

where $X'_{(1)} \in T$. Hence using lemma 3.5 $\Lambda_m(T) * U_q(g) = U_q(g) * \Lambda_m(T)$ $\forall m \in \mathbb{N}^0$. Since $\Lambda(T) = \bigoplus_{m=0}^{\infty} \Lambda_m(T)$ and $\Lambda_m(T) * \Lambda_p(T) \subset \Lambda_{m+p}(T)$ the statement follows easily with the help of lemma 2.4.2.

Ad 2. The consistency of the definition of γ is obviously since $\gamma_{|U_q(g)} = \mathrm{id}_{U_q(g)}$ and the algebra relations are grade preserving according to the graded space $\Lambda(T)$. Then bijectivity and uniqueness can be deduced immediately. □

Now everything needed is provided to state

Theorem 4.2. Let g be a Lie algebra of type A_n, B_n, C_n or D_n and let $K(q, g)$ be the corresponding algebra of adjoint differential operators according to (4.1). Then there exists a unique exterior derivative

$$\mathrm{d} : K(q, g) \to K(q, g)$$

such that

(i) d is $U_q(g)$-module morphism.

(ii)

$$\begin{aligned} &\mathrm{d}\,(X) = \tilde{X} \qquad \forall X \in T \subset \Lambda(T) \\ &\mathrm{d}\,(k * l) = \mathrm{d}\,(k) * l + \gamma\,(k) * \mathrm{d}\,(b) \qquad \forall k,\ b \in K(q, g)\,. \end{aligned} \tag{4.3}$$

Proof. The consistency of properties (i) and (ii) must be checked only on relations which completely determine the algebra $K(q, g)$ since γ and d (by definition) preserve associativity and the relations. Because of lemma 2.4 there are three kinds of essential algebraic relations:

(i) Pure $U_q(g)$-relations which are satisfied automatically as $\mathrm{d}_{|U_q(g)} = 0$.

(ii) Pure $\Lambda(T)$-relations.

(iii) Mixed relations given by (2.16).

For the calculations we restrict to A_n. The results for B_n, C_n and D_n are similar. The consistency of the definition of d with the mixed relations (2.16) is easily verified if we use the basis $\{\tilde{X}^{a[b]}\} \subset \tilde{T} \subset U_q(g)$ and $\{X^{c[d]}\} \subset T \subset \Lambda(T)$ and exploit equations (3.5) and (3.10):

$$\begin{aligned} \mathrm{d}\left(\tilde{X}^{k[l]}\, X^{i[j]} - \mathrm{ad}^\wedge(\tilde{X}^{k[l]})(X^{i[j]}) - \mathrm{ad}^\wedge({\Theta^k}_{[r]}{}^{[l]}{}_s)(X^{i[j]})\, X^{s[r]}\right) &\\ = \tilde{X}^{k[l]}\, \tilde{X}^{i[j]} - [\tilde{X}^{k[l]},\ \tilde{X}^{i[j]}] - \hat{\sigma}^{[j]i[l]k}{}_{[r]s[u]v}\, \tilde{X}^{v[u]}\, \tilde{X}^{s[r]} &\\ = 0\,. & \end{aligned}$$

Finally, let $y = A_{[j]i[k]l}\, X^{l[k]} \otimes X^{i[j]}$ be an arbitrary element in $\mathrm{Ker}(\mathrm{id}_{T^{\otimes 2}} - \sigma)$ or equivalently stated let A be an arbitrary vector in $\mathrm{Ker}(\mathbf{1} - \hat{\sigma}^T)$. Then we have $\bar{y} = 0 \in \Lambda(T)$. Making use of the relations (2.16) the application of d yields

$$\begin{aligned} \mathrm{d}(\bar{y}) =& A_{[j]i[k]l}\left(\tilde{X}^{l[k]} * X^{i[j]} - X^{l[k]} * \tilde{X}^{i[j]}\right) \\ =& A_{[j]i[k]l}\left(\frac{1}{\lambda}\left(C^{[l]k}\, \delta^i_v\, \delta^{[j]}_{[u]} - \hat{\sigma}^{[j]i[l]k}{}_{[a]b[u]v}\, C^{[a]b}\right) X^{v[u]}\right. \\ & \left. - (\mathbf{1} - \hat{\sigma})^{[j]i[l]k}{}_{[a]b[u]v}\, X^{v[u]}\, \bar{X}^{b[a]}\right). \end{aligned}$$

For the matrix $\hat{\sigma}$ the folowing identity holds:

$$\hat{\sigma}^{[j]i[l]k}{}_{[a]b[u]v}\, C^{[a]b} = (\hat{\sigma}^2)^{[j]i[l]k}{}_{[u]v[a]b}\, C^{[a]b} \tag{4.4}$$

which can be verified immediately if we use graphical notation and apply some $(\varepsilon, \hat{R}_q)$-relations. Hence

$$\mathrm{d}(\bar{y}) = A_{[j]i[k]l}(\mathbf{1} - \hat{\sigma})^{[j]i[l]k}{}_{[a]b[u]v}\left(\frac{1}{\lambda}\, (\mathbf{1} + \hat{\sigma})^{[a]b[u]v}{}_{[c]d[e]f}\, C^{[e]f}\, X^{d[c]} - X^{v[u]}\, \bar{X}^{b[a]}\right) = 0$$

since $A \in \mathrm{Ker}(\mathbf{1} - \hat{\sigma}^T)$. Conversely if $\mathrm{d}(\bar{y}') = 0$ where $y' = A'_{[j]i[k]l}\, X^{l[k]} \otimes X^{i[j]}$, it follows that $A' \in \mathrm{Ker}(\mathbf{1} - \hat{\sigma}^T)$. □

A straightforward consequence of theorem 4.2 is

Corollary 4.3. The derivative d is nilpotent and anticommutes with γ. It maps $K_m(q, g)$ into $K_{m-1}(q, g)$.

$$\begin{aligned} &\mathrm{d}^2 = 0 \\ &\mathrm{d}\gamma + \gamma\mathrm{d} = 0 \\ &\mathrm{d}(K_m(q, g)) \subset K_{m-1}(q, g)\,. \end{aligned} \tag{4.5}$$

Hence $(K(q, g), \mathrm{d})$ is a complex. □

As the derivative d is a $U_q(g)$-module morphism we can construct cohomologies of the quantum enveloping algebra $U_q(g)$ which are deformations of the classical cohomologies of Lie algebras. For that purpose we consider an arbitrary $U_q(g)$-module $\mathcal{M}$. Since $K_m(q, g)$ $\forall m \in \mathbb{N}^0$ is a $U_q(g)$-module we can define the vector space of $U_q(g)$-module morphisms

$$L(q, g) = \bigoplus_{m=0}^{\infty} L_m(q, g) := \bigoplus_{m=0}^{\infty} \mathrm{Hom}_{U_q(g)}(K_m(q, g), \mathcal{M}) . \tag{4.6}$$

Now let $\mathrm{d}_m := \mathrm{d}_{|K_m(q,g)}$ $\forall m \in \mathbb{N}^0$. With the help of the usual arguments one concludes that the mapping $\delta_m := \cdot \circ \mathrm{d}_{m+1}$ $\forall m \in \mathbb{N}^0$ is a $U_q(g)$-module morphism which maps $L_m(q, g)$ to $L_{m+1}(q, g)$. Denote the coboundary operator by $\delta := \bigoplus_{m=0}^{\infty} \delta_m$. This yields $\delta^2 = 0$ and thus $(L(q, g), \delta)$ is the desired (cochain) complex which defines cohomology groups of $U_q(g)$ relative to the $U_q(g)$-module $\mathcal{M}$. The obtained cohomology structure is a deformation of the Koszul cohomology for Lie algebras. However, there is a difference between the classical and the quantum result since the quantized sequence

$$\mathbb{C} \xleftarrow{\epsilon} K_0(q, g) = U_q(g) \xleftarrow{\mathrm{d}_1} K_1(q, g) \xleftarrow{\mathrm{d}_2} K_2(q, g) \xleftarrow{\mathrm{d}_3} \cdots \tag{4.7}$$

is no more exact. The reason is that in the quantum case we have established $(n+1)^2$ linearly independent vector fields for A_n instead of $(n+1)^2 - 1$. Similar things happen for B_n, C_n and D_n. The required dimensional restriction in the quantum enveloping algebra $U_q(g)$ takes place in higher-order relations, namely in the invertibility relations mentioned in the appendix. To verify the non-exactness of the sequence (4.7) we consider for simplicity the case A_1 where the additional contracted relation in $U_q(g)$ is given by

$$D_{jkda}\, \tilde{X}^{ad}\, \tilde{X}^{kj} + 2\, q^{-1}\, \varepsilon_{jk}\, \tilde{X}^{kj} = 0 \tag{4.8}$$

here D is a matrix formed of $\hat{R}_q$ and ε. Thus the element

$$z(X, \tilde{X}) := D_{jkda}\, X^{ad} * \tilde{X}^{kj} + 2\, q^{-1}\, \varepsilon_{jk}\, X^{kj} \tag{4.9}$$

is closed, $\mathrm{d}(z) = 0$. Now suppose z is exact, i.e. $\exists y(X, \tilde{X}) \in K_2(q, g)$ of the form

$$y(X, \tilde{X}) = X^{kj}\, X^{ba}\, f(\tilde{X})_{abjk} \tag{4.10}$$

such that $\mathrm{d}y = z$. Using the linear independence of the set $\{(X^{ij})_{i,j}\}$ and lemma 2.4.3 one obtains

$$\frac{1}{\lambda}\, (\mathbf{1} - \hat{\sigma}^2)^{jilk}{}_{abuv}\, \varepsilon^{uv}\, \epsilon(f(\tilde{X})_{jilk}) = 2\, q^{-1}\, \varepsilon_{ab} . \tag{4.11}$$

If we contract in equation (4.11) the free indices with ε^{ab} and employ the identity $(\hat{\sigma}^2)^{jilk}{}_{abuv}\, \varepsilon^{ab}\, \varepsilon^{uv} = \varepsilon^{ji}\, \varepsilon^{lk}$ we arrive at $0 = 2\,(q + q^{-1})$. This is a contradiction and it is thus proven that the sequence (4.7) is not exact. In a very similar way the non-exactness is shown for all A_n and for B_n, C_n and D_n. In the case of A_n it is possible to show that the factorization $z(X, \tilde{X}) = 0$ is not compatible with the exterior product. Thus one cannot use relation (4.9) for a dimensional reduction of the exterior algebra. Therefore the element (4.9) destroys the exactness of the sequence.

In the remaining part of our work we restrict ourselves to the series A_n. We sketch some A_n-specific facts without going into detail.

(i) For A_n there exists essentially one invertibility relation which controls the required dimensional restriction in $U_q(g)$. One can thus prove that only one integrated one-form (relative to the grading in $K(q,g)$) is responsible for the destruction of the exactness in the space $K_1(q,g)$.

(ii) We construct an algebra $K'(q,g)$ generated by vector fields $(\tilde{X}'^{i[j]})_{i,[j]}$ and by one-forms $(X'^{i[j]})_{i,[j]}$ with relations

$$(\mathbf{1}-\hat{\sigma})^{[l]k[j]i}{}_{[b]a[c]d}\,\tilde{X}'^{d[c]}\,\tilde{X}'^{a[b]} = \frac{1}{\lambda}\,(\mathbf{1}-\hat{\sigma}^2)^{[l]k[j]i}{}_{[b]a[c]d}\,C^{[c]d}\,\tilde{X}'^{a[b]}$$

$$\begin{aligned}\tilde{X}'^{i[j]}\,X'^{k[l]} - \hat{\sigma}^{[l]k[j]i}{}_{[b]a[c]d}\,X'^{d[c]}\,\tilde{X}'^{a[b]} \\ = \frac{1}{\lambda}\,(\mathbf{1}-\hat{\sigma}^2)^{[l]k[j]i}{}_{[b]a[c]d}\,C^{[c]d}\,X'^{a[b]}\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&A_{[i]j[k]l}\,X'^{l[k]}\,X'^{j[i]} = 0\\ \Longleftrightarrow\ &A \in \mathrm{Ker}(\mathbf{1}-\hat{\sigma}^T)\end{aligned} \tag{4.13}$$

where we use the notation and the $\hat{\sigma}$-matrix of A_n according to (3.8). Denote by $\tilde{T}' := [(\tilde{X}'^{i[j]})_{i,[j]}]_{\mathbb{C}} \subset K'(q,g)$ the vector space of vector fields and by $\Lambda'_m \subset K'(q,g)$ the vector space of monomials in $(X'^{i[j]})_{i,[j]}$ of degree $m \in \mathbb{N}^0$. Then a proposition analogous to proposition 4.1 holds if we replace $K(q,g)$ by $K'(q,g)$, $K_m(q,g)$ by $K'_m(q,g)$, $\Lambda_m(T)$ by Λ'_m, $U_q(g)$ by $K'_0(q,g)$ and γ by γ'. Likewise in theorem 4.2 a derivative d' can be established on $K'(q,g)$ such that d' is a $K'_0(q,g)$-module morphism and has corresponding properties as d in (4.3).

Now for the complex $(K'(q,g),\mathrm{d}')$ the exactness can be conjectured (at least for $q \neq$ root of unity). We proceed in a similar way as in [Jac].

We define the space

$$\tilde{T}'^{(j)} := \sum_{i=0}^{j} \tilde{T}'^{i} \tag{4.14}$$

where $\tilde{T}'^0 = \mathbb{C}$ and the space

$$K'^{(j)}(q,g) := \sum_{h+k\leqslant j} \Lambda'_k\,\tilde{T}'^{(h)}\,. \tag{4.15}$$

It can be shown that

$$K'^{(j)}(q,g) := \bigoplus_{m=0}^{j} \Lambda'_m\,\tilde{T}'^{(j-m)}\,. \tag{4.16}$$

Since $(K'^{(j)}(q,g))_{j\in\mathbb{N}^0}$ is a filtration of $K'(q,g)$ we are able to define the associated graded algebra $K_c{}'(q,g)$ [Jac].

$$\begin{aligned}&K_c{}'(q,g) := \bigoplus_{m=0}^{\infty} K_c{}'_m(q,g)\\ &K_c{}'_m(q,g) := K'^{(m)}(q,g) \,\big/\, K'^{(m-1)}(q,g)\\ &K'^{(-1)}(q,g) := \{0\}\,.\end{aligned} \tag{4.17}$$

This algebra can be regarded as the q-commutative analogue of the Koszul complex of a certain commutative Lie algebra. Since $K'^{(m)}(q,g)$ is a subcomplex of $K'(q,g)$ it follows that $({K_c}'_m(q,g),\ \bar{\mathrm{d}}^m := \overline{\mathrm{d}_{|K'^{(m)}(q,g)}})$ is a complex, the so-called mth difference complex. If it is possible to prove that $({K_c}'_m(q,g),\ \bar{\mathrm{d}}^m)\ \forall m \in \mathbb{N}$ is exact one deduces like in [Jac] the exactness of $(K'(q,g),\ \mathrm{d}')$. The algebra ${K_c}'(q,g)$ can be identified as the algebra generated by the generators $(T^{i[j]})_{i,[j]}$ and $(\tilde{T}^{i[j]})_{i,[j]}$ which obey the relations

$$\begin{aligned}&(\mathbf{1}-\hat{\sigma})^{[l]k[j]i}{}_{[b]a[c]d}\,\tilde{T}^{d[c]}\,\tilde{T}^{a[b]} = 0\\ &(\mathbf{1}-\hat{\sigma})^{[l]k[j]i}{}_{[b]a[c]d}\tilde{T}^{i[j]}\,T^{k[l]} - \hat{\sigma}^{[l]k[j]i}{}_{[b]a[c]d}\,T^{d[c]}\,\tilde{T}^{a[b]} = 0\end{aligned}\tag{4.18}$$

and

$$\begin{aligned}&A_{[i]j[k]l}\,T^{l[k]}\,T^{j[i]} = 0\\ \Longleftrightarrow\ &A \in \mathrm{Ker}(\mathbf{1}-\hat{\sigma}^T)\,.\end{aligned}\tag{4.19}$$

Then the subspaces ${K_c}'_m(q,g)$ are the spaces of monomials in $(T^{i[j]},\ \tilde{T}^{k[l]})_{i,[j],k,[l]}$ of total degree m. To prove the exactness of $({K_c}'_m(q,g),\ \bar{\mathrm{d}}^m)\ \forall m \in \mathbb{N}$ it suffices to find a linear map D_m in ${K_c}'_m(q,g)$ such that

$$D_m\,\bar{\mathrm{d}}^m + \bar{\mathrm{d}}^m\,D_m = \mathrm{id}_{{K_c}'_m(q,g)}\,. \tag{4.20}$$

We abbreviate formally the tensor element $t_{r,s} := T^{i_1[j_1]}\cdot\ldots\cdot T^{i_r[j_r]}\,\tilde{T}^{k_1[l_1]}\cdot\ldots\cdot\tilde{T}^{k_s[l_s]}$. The matrix $\tilde{\sigma}$ is defined through

$$\tilde{\sigma}^{[i]j[k]l}{}_{[m]n[o]p} := \hat{\sigma}^{[k]l[i]j}{}_{[o]p[m]n} \tag{4.21}$$

and $\tilde{\sigma}_l := (\mathbf{1}_{as}\otimes_{(1.)}\cdots\otimes_{(l-1).}\tilde{\sigma}\otimes_{(l+1).}\cdots\otimes_{(m).}\mathbf{1}_{as})$ where $\mathbf{1}_{as} = (P_n)$ (see definition (A.11)). Then we get for $r \geqslant 2$

$$\bar{\mathrm{d}}^{r+s}(t_{r,s}) = \left[\sum_{k=0}^{r-1}(\tilde{\sigma}_{k+1}\cdot\ldots\cdot\tilde{\sigma}_{r-1})\right]t_{r-1,s+1}\,. \tag{4.22}$$

For $r \in \{0,1\}$ the results are obvious. Now we make the ansatz

$$D_{r+s}(t_{r,s}) := \lambda^{r+s}_{r,s}\,t_{r+1,s-1} \tag{4.23}$$

with $\lambda^{r+s}_{r,s}$ a matrix which has to be determined. Equation (4.20) yields

$$\lambda^r_{r,0} = 0 \tag{4.24}$$

$$(\lambda^s_{0,s}-\mathbf{1})\,t_{0,s} = 0 \tag{4.25}$$

$$\left(\lambda^{s+1}_{1,s}\,(\tilde{\sigma}_1-\mathbf{1}) + \lambda^{s+1}_{0,s+1}-\mathbf{1}\right)t_{1,s} = 0 \tag{4.26}$$

and

$$\left(\sum_{k=0}^{r-1}(-1)^k\,(\tilde{\sigma}_{k+1}\cdot\ldots\cdot\tilde{\sigma}_{r-1})\,\lambda^{r+s}_{r-1,s+1} + \lambda^{r+s}_{r,s}\,\sum_{k=0}^{r}(-1)^k\,(\tilde{\sigma}_{k+1}\cdot\ldots\cdot\tilde{\sigma}_r)-\mathbf{1}\right)t_{r,s} = 0 \tag{4.27}$$

for $r \geqslant 2$. The representation theories of $U(g)$ and of $U_q(g)$ for $\{q \neq \text{root of unity}\}$ are equivalent [Res], the matrix $\hat{\sigma}$ admits a complete projector decomposition according to proposition 3.3 and the projector $P_1(\hat{\sigma})$ in (3.17) has the form $P_1(\hat{\sigma}) = (\tilde{\mathcal{P}}^{(+)}, \mathcal{P}^{(+)}) + (\tilde{\mathcal{P}}^{(-)}, \mathcal{P}^{(-)})$ (for notation see [CSWW, FRT]). For $r = 0$ we have the additional condition

$$(\mathbf{1} - \tilde{\sigma}_l)\, \lambda^s_{0,s}\, t_{1,s-1} = 0 \forall l \in \{1, \ldots, s-1\} \tag{4.28}$$

which arises from the application of D_m to the identity $(\mathbf{1} - \tilde{\sigma}_l)\, t_{0,s} = 0$. For $s = 2$ we obtain $\lambda^2_{0,2} = P_1(\tilde{\sigma})$. Then for $s > 2$ we can construct a totally q-symmetric tensor similar as in [HW]. One has to use the matrix

$$\sigma' := q^2\, (\tilde{\mathcal{P}}^{(+)}, \mathcal{P}^{(+)}) + q^{-2}\, (\tilde{\mathcal{P}}^{(-)}, \mathcal{P}^{(-)}) - (\tilde{\mathcal{P}}^{(+)}, \mathcal{P}^{(-)}) - (\tilde{\mathcal{P}}^{(-)}, \mathcal{P}^{(+)}) \tag{4.29}$$

as a braid matrix. Hence we conjecture that $\lambda^s_{0,s} :=$ {totally q-symmetric projector in the s index pairs $(i_a[j_a])_{a=1,\ldots,s}$} exists in the sth tensor representation corresponding to the matrix $\hat{\sigma}$ (or $\tilde{\sigma}$), that for these representations a complete projector decomposition is possible and then there should exist a covariant solution of the $\lambda^{r+s}_{r,s}$ such that the matrix coefficients in (4.26) and (4.27) vanish independently of $t_{r,s}$. All this yields the exactness of the complex $(K'(q, g),\ \mathrm{d}')$.

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Appendix

The quantum enveloping algebra $U_q(g)$, $q \in \mathbb{C} \setminus \{1\}$ for Lie algebras g of type A_n, B_n, C_n and D_n is Hopf algebra isomorphic to a certain minimal completion of the Hopf algebra of regular functionals of the corresponding quantum group [FRT]. $U_q(g)$ is then generated by the regular functionals $(L^{\pm i}{}_j)_{i,j=1,\ldots,N}$ ($N = n+1$ for A_n, $N = 2n+1$ for B_n and $N = 2n$ for C_n, D_n) which in particular obey the Hopf relations

$$\begin{aligned}
&\Delta(L^{\pm i}{}_j) := L^{\pm i}{}_k \otimes L^{\pm k}{}_j \\
&\epsilon(L^{\pm i}{}_j) := \delta^i_j \\
&S(L^{\pm i}{}_j) = \begin{cases} \dfrac{q^{-\binom{n+1}{2}}}{[n]_q!}\, \varepsilon^{k_n \ldots k_1 i}\, L^{\pm l_1}{}_{k_1} \ldots L^{\pm l_n}{}_{k_n}\, \varepsilon_{j\, l_n \ldots l_1} & \text{for} \quad A_n \\ C^{ki}\, L^{\pm l}{}_k\, C_{jl} & \text{for} \quad B_n,\ C_n,\ D_n \end{cases}
\end{aligned} \tag{A.1}$$

and the commutator relations

$$\begin{aligned}
\hat{R}_q{}^{ji}{}_{lk}\, L^{\pm k}{}_v\, L^{\pm l}{}_w &= L^{\pm i}{}_a\, L^{\pm j}{}_b\, \hat{R}_q{}^{ba}{}_{wv} \\
\hat{R}_q{}^{ji}{}_{lk}\, L^{+k}{}_v\, L^{-l}{}_w &= L^{-i}{}_a\, L^{+j}{}_b\, \hat{R}_q{}^{ba}{}_{wv}
\end{aligned} \tag{A.2}$$

where Δ, ϵ and S are the comultiplication, counit and antipode of $U_q(g)$, respectively, $\varepsilon_{i_1\ldots i_{n+1}} = (-1)^n \cdot \varepsilon^{i_1\ldots i_{n+1}} = (-q)^{l(\sigma)}$ if $l(\sigma)$ is the minimal number of transpositions of the permutation $\sigma = \binom{1\ \ldots\ n+1}{i_1\ \ldots\ i_{n+1}}$, $\varepsilon_{i_1\ldots i_{n+1}} = \varepsilon^{i_1\ldots i_{n+1}} = 0$ else, $[n]_q!$ is the usual q-factorial

[CSWW] and C_{ij} is the usual q-metric [FRT]. The $\hat{R}_q$-matrices for the respective quantum groups are taken from [FRT]. We use a summation rule throughout the paper if not otherwise mentioned.

The $\hat{R}_q$-matrices, the metric C and the ε-tensor obey various useful relations [CSWW, FRT, Res]. For instance we have a projector equation for A_n

$$\begin{aligned}\varepsilon_{j_1\ldots j_l\,k_{l+1}\ldots k_{n+1}}\,\varepsilon^{k_{l+1}\ldots k_{n+1}\,i_1\ldots i_l} &= \varepsilon^{i_1\ldots i_l\,k_{l+1}\ldots k_{n+1}}\,\varepsilon_{k_{l+1}\ldots k_{n+1}\,j_1\ldots j_l}\\ &= (-1)^{n(l-1)}\,q^{\binom{n+1}{2}}[n+1-l]_q![l]_q!\,(P_l)^{i_1\ldots i_l}{}_{j_1\ldots j_l}\,.\end{aligned} \tag{A.3}$$

(P_l) is the projector to the lth-order q-antisymmetric tensor representation.

The vector fields defined in [CDSWZ]

$$X^i{}_j = \frac{1}{\lambda}\,(\delta^i{}_i\,\mathbf{1}_{U_q(g)} - \Theta^{ik}{}_{jk}) \tag{A.4}$$

have a comultiplication

$$\Delta(X^i{}_j) = X^i{}_j \otimes \mathbf{1}_{U_q(g)} + \Theta^{ik}{}_{jl} \otimes X^l{}_k \tag{A.5}$$

where $\Theta^{ik}{}_{jl} = L^{+i}{}_l S(L^{-k}{}_j)$ and $\lambda = (q - q^{-1})$. Additionally the vector fields satisfy some kind of invertibility and commutation relations [DJSWZ] which can be deduced from the identities (A.1) and (A.2). The set $\{X^i{}_j\}_{i,j}$ is linearly independent† in $U_q(g)$. The vector space of vector fields will be denoted by

$$T := [X^i{}_j \mid X^i{}_j \in U_q(g)]_{\mathbb{C}}\,. \tag{A.6}$$

In [Bur, DJSWZ] it is shown that T generates $U_q(g)$ up to a certain completion. Similar to [CSWW] we preform linear transformations with the matrices

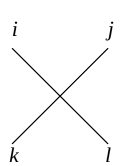
$$C^{[j]i} := \varepsilon^{j_1\ldots j_n i} \qquad C_{i[j]} := \frac{q^{-\binom{n+1}{2}}}{[n]_q!}\,\varepsilon_{i\,j_1\ldots j_n} \tag{A.7}$$

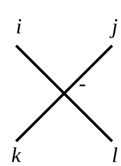
according to

$$F_i \mapsto F^{[j]} := C^{[j]i}\,F_i \qquad F^i \mapsto F_{[j]} := F^i\,C_{i[j]} \tag{A.8}$$

for A_n and analogously for B_n, C_n and D_n with the metric C^{ij} and C_{ij}.

To simplify expressions we use graphical notation [CSWW, Res]. The fundamental quantities are denoted by

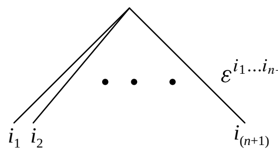
$\hat{R}_q^{ij}{}_{kl} =$ 

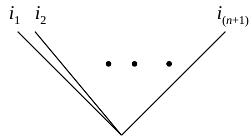
$\hat{R}_q^{-1ij}{}_{kl} =$ 

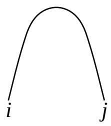
(A.9)

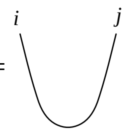
† This is true if some root-of-unity values for q are excluded, which will be assumed henceforth.

(graphical presentation of the R-matrix $\hat{R}$ and of the inverse of the R-matrix $\hat{R}$)

$\varepsilon_{i_1\ldots i_{n+1}} =$ 

$\varepsilon^{i_1\ldots i_{n+1}} =$ 

$C_{ij} =$ 

$C^{ij} =$ 

(A.10)

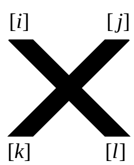
(graphical presentations of the covariant q-ε-tensor in the case A_n, of the contravariant q-ε-tensor in the case A_n, of the covariant q-metric in the case B_n, C_n and D_n, and of the contravariant q-metric in the case B_n, C_n and D_n).

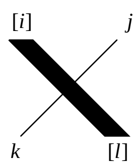
For A_n the projector (P_n) will be represented graphically.

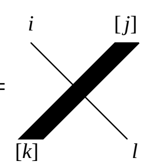
$(P_n)^{[i]}{}_{[j]} =$ 

(A.11)

(graphical presentation of the projector P_n on the q-antisymmetric n-tensors in the case A_n). The notation

$\hat{R}_q^{[i][j]}{}_{[k][l]} =$ 

$\hat{R}_q^{[i]j}{}_{k[l]} =$ 

$\hat{R}_q^{i[j]}{}_{[k]l} =$ 

(A.12)

(graphical presentations of the R-matrix corresponding to the braiding of two copies of the space of q-antisymmetric n-tensors in the case A_n, of the R-matrix corresponding to the braiding of the space of q-antisymmetric n-tensors with the canonical vector space representation in the case A_n, and of the R-matrix corresponding to the braiding of the canonical vector space representation with the space of q-antisymmetric n-tensors in the case A_n) is then evident. The inverse matrices have analogue structure. Since the matrix $\hat{R}_q^{ij}{}_{kl}$ satisfies the Hecke equation

$$(\hat{R} - q\,\mathbf{1})(\hat{R} + q^{-1}\mathbf{1}) = 0 \tag{A.13}$$

the matrix $\hat{\mathcal{R}}_q^{[i][j]}{}_{[k][l]} = q^{-(n-1)}\,\hat{R}_q^{[i][j]}{}_{[k][l]}$ obeys the same relation. In the cases of B_n, C_n and D_n the matrix $\hat{R}_q$ enters in representations of the BWM algebra [FRT].

$$(\hat{R}_q - q\,\mathbf{1})(\hat{R}_q + q^{-1}\mathbf{1})(\hat{R}_q - \varepsilon\, q^{\varepsilon - N}\,\mathbf{1}) = 0 \tag{A.14}$$

where $\varepsilon = 1$ for B_n, D_n and $\varepsilon = -1$ for C_n.

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